UNIT 1

The Straight Line

I. INTRODUCTORY

Geometry is the science of space and deals with the shapes, sizes and positions of things.

Euclid was a Greek mathematician of the third century B.C. who wrote a remarkable book called *The Elements.* For 2000 years, the first six books of this work have been used as an introduction to Geometry. Modern textbooks contain in a modified form the subject matter of Books I–IV, VI and XI of Euclid's work.

Axioms. All reasoning is based on certain elementary statements, the truth of which is admitted without discussion: such statements are called axioms.

Euclid gave a list of twelve axioms: of these, the following relate to magnitudes of all kinds*: axioms relating to geometrical magnitudes will be stated later.

1. Things which are equal to the same thing are equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are taken from equals, the remainders are equal.
4. Doubles of equal things are equal.
5. Halves of equal things are equal.
6. The whole is greater than its part.
7. If equals are added to unequals, the wholes are unequal.
8. If equals are taken from unequals, the remainders are unequal.

Surface, Line, Point. The space occupied by any object (say a brick) is limited by boundaries which separate it from surrounding space. These boundaries are called **surfaces**.

DEFINITION. A **surface** has length and breadth but no thickness.

Surfaces meet (or intersect) in *lines.* For example, a wall of a room meets the floor in a straight line.

DEF. A **line** has length but no breadth. Lines meet (or intersect) in *points.*

* The student should read these through in order to get a clear idea of what kinds of truths are assumed as axiomatic. It is quite unnecessary to commit them to memory.

Def. A **point** has position but no magnitude.

In practice, the mark traced by a pencil-point on a sheet of paper is called a line. But it is not a line according to the definition: for, however thin it may be, it has some breadth.

Again, if we make a dot on the paper as a mark of position, the dot is not a geometrical point, for it has some magnitude.

Straight Lines. Lines are either *straight* or *curved.* Everyone knows what is meant by a *straight line*, but it is difficult to put the exact meaning into words. Euclid's definition was as follows:

Def. A straight line is that which lies evenly between its extreme points.

To this he added the following:

Axiom. Two straight lines cannot enclose a space.

Postulate* 1. Let it be granted that a straight line may be drawn from any one point to any other point.

Postulate 2. Let it be granted that a straight line may be produced to any length in a straight line.

Hence it follows that

(*i*) Two straight lines cannot intersect in more than one point. For, if they met in two points, they would enclose a space.

(*ii*) One and only one straight line can be drawn between two given points. When we draw this straight line, we are said to join the points.

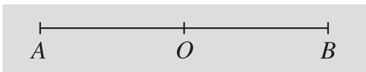
In order to compare two straight lines, we may suppose one of them to be taken up and placed on the other. If the lines fit exactly, they are said to *coincide.*

Def. *Equal straight lines* are those which can be made to coincide.

Equal straight lines are said to have the same length.

The distance between two points is the length of the straight line joining them.

If O is a point in a straight line AB, lengths measured from O towards B are said to be measured in the opposite sense to those measured from O towards A.



Def. Any part of a straight line is called a **segment** of that straight line.

The Plane. In order to find out whether the surface of a board is what is called 'plane' or not, a carpenter applies a straight edge (the straight edge of his plane for instance) to the board, *in all directions*; *i.e.* he practically joins *any* two points in the surface with a straight line and sees whether this straight line is in contact with the board throughout its length.

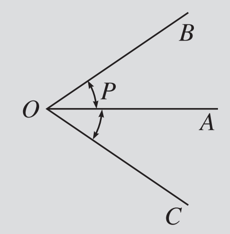
*A postulate is a *demand.*

Def. A **plane** is a surface in which any two points being taken the straight line between them lies wholly in that surface.

A plane is often called a *plane surface.*

Angles. Def. Two straight lines, drawn from the same point, are said to contain an angle.

The straight lines are called the arms of the angle; their point of intersection is called the vertex of the angle.



The angle contained by two straight lines *OA* and *OB* is called 'the angle *AOB*' or 'the angle *BOA*,' or, *if there is only one angle at the point*, simply 'the angle *O*.'

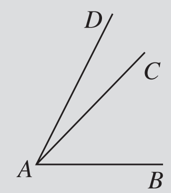
If a straight line *OP*, starting from the position *OA*, revolves about the point *O* as a hinge, until it reaches the position *OB*, it is said to turn through or describe the angle *AOB*.

In order to describe the angle *AOC* in the above figure, the revolving line would have to turn in the opposite direction or sense to that in which it turned to describe the angle *AOB*.

We therefore say that the angles *AOB*, *AOC* are described **in opposite senses**.

Def. Two angles are said to be **equal** when one of them can be placed so that its arms fall along the arms of the other.

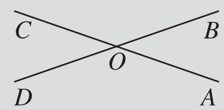
Note that *the size of an angle does not depend on the lengths of its arms.*



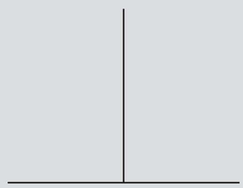
Def. Two angles (*BAC*, *CAD*), which are situated on opposite sides of a common arm and have a common vertex, are called **adjacent angles**.

The angle *BAD* is the sum of the angles *BAC*, *CAD*. The angle *CAD* is the difference of the angles *BAD*, *BAC*.

The bisector of an angle is the straight line which bisects it, *i.e.*, which divides it into two equal parts.



Def. If the arms of an angle *AOB* are produced through *O* to *C* and *D*, the angles *AOB*, *COD* are said to be **vertically opposite**: so also are the angles *BOC*, *DOA*.

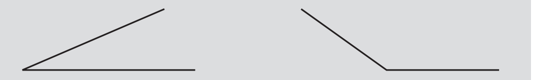


Def. When one straight line, standing on another straight line, makes the adjacent angles equal, each of these angles is called a right angle; and the straight line standing on the other is said to be **perpendicular** to it.

Axiom. All right angles are equal.

DEF. The straight line which bisects a given straight line, and is perpendicular to it, is called the **perpendicular*** bisector of the line.

The distance of a point from a straight line is the length of the perpendicular drawn from the point to the line.

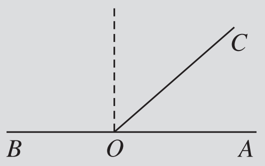


Acute angle Obtuse angle

DEF. An **acute angle** is an angle which is less than a right angle.

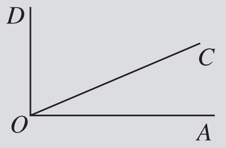
DEF. An **obtuse angle** is an angle which is greater than a right angle.

DEE. Two angles are said to be **supplementary** when their sum is two right angles. In this case, each angle is called the supplement of the other.



Thus, the angles *AOC*, *COB* are **supplementary**.

DEF. Two angles are said to be **complementary** when their sum is one right angle. In this case, each angle is called the complement of the other.



Thus, the angles *AOC*, *COD* are complementary.

The table of angular measurement is as follows:

1 right angle = 90 degrees (90°)

1 degree = 60 minutes (60′)

1 minute = 60 seconds (60″).

Plane Geometry deals with the properties of lines and points which are in the same plane.

The discussion is divided into propositions which are either theorems or constructions.

A **theorem** is a proposition in which some statement has to be proved.

The statement itself is called the **enunciation**.

The enunciation of a theorem consists of two parts. The first part, called the **hypothesis**, states what is assumed. The second part, called the conclusion, states what has to be proved.

For example, a simple theorem in Algebra is as follows:

If $a = b$, then $na = nb$.

The hypothesis is $a = b$; the conclusion is $na = nb$.

If two theorems are such that the hypothesis of each is the conclusion of the other, then either of the theorems is said to be the converse of the other.

Thus, the converse of the above theorem is

If $na = nb$, *then* $a = b$.

*Sometimes also it is called the **right bisector**.

Note particularly that the converse of a theorem is not necessarily true.

Thus, consider the theorem *If* $a = b$, *then* $a^2 = b^2$.

The converse is *If* $a^2 = b^2$, *then* $a = b$, and this is not correct, for *if* $a^2 = b^2$ we can only conclude that *either* $a = b$ or $a = -b$.

When a theorem has been proved, it is sometimes found that other important theorems follow so easily that they hardly require formal proofs.

Such theorems are called corollaries.

> **NOTE:** In writing out Theorems, an accurate figure drawn with instruments is not expected. A neat freehand drawing is sufficient.

A **construction** is a proposition in which it is required to draw some particular figure.

It is convenient to arrange the theorems and constructions in separate groups.

In order to do this we make the following Postulates:

Let it be granted that

1. From the greater of two given straight lines, a part can be cut off equal to the less.
2. A straight line may be bisected (that is divided into two equal parts).
3. At any point in a given straight line, a perpendicular can be drawn to that straight line.
4. At any point in a given straight line, a straight line can be drawn, making with the given line an angle equal to a given angle.

Other postulates of this kind will be stated further on.

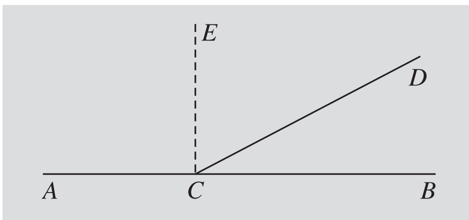
These postulates are sometimes called **hypothetical constructions**.

> **NOTE:** *It is suggested that, in a first course, the formal treatment of Theorems* 1, 2, 3, 4, 6, 7, 8, 9, 10, 14, 17, 18 *should be replaced by oral explanations, the facts stated in the enunciations being regarded as axiomatic.*

II. ANGLES AT A POINT

THEOREM 1 (Euclid I. 13)

If a straight line stands on another straight line the sum of the two adjacent angles is two right angles.



Let the straight line *CD* stand on the straight line *ACB*.

It is required to prove that

$$\angle ACD + \angle DCB = 2 \text{ right angles.}$$

Construction. Let *CE* be drawn perpendicular to *AB*.

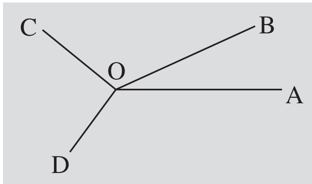
Proof. $\angle ACD + \angle DCB = \angle ACE + \angle ECD + \angle DCB.$

Also $\angle ACE + \angle ECB = \angle ACE + \angle ECD + \angle DCB.$

$\therefore$ $\angle ACD + \angle DCB = \angle ACE + \angle ECB.$

But, by construction, $\angle$s *ACE*, *ECB* are right angles.

$\therefore$ $\angle ACD + \angle DCB = 2$ right angles.



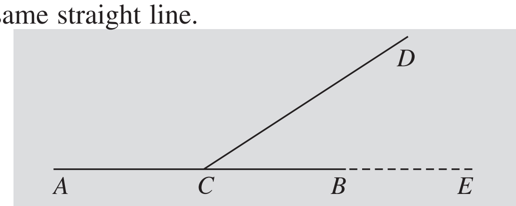
COROLLARY. If any number of straight lines are drawn from a given point, the sum of the consecutive angles so formed is four right angles.

Thus, in the figure,

$$\angle AOB + \angle BOC + \angle COD + \angle DOA = 4 \text{ right angles.}$$

THEOREM 2* (Euclid I. 14)

If at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in the same straight line.



At the point *C*, in the straight line *CD*, let the straight lines *CA*, *CB*, on opposite sides of *CD*, make the adjacent angles *ACD*, *DCB* together equal to two right angles.

It is required to prove that *CA*, *CB* are in the same straight line.

Construction. Produce *AC* to some point *E*.

Proof. By construction, *ACE* is a straight line;

$\therefore$ $\angle ACD + \angle DCE = 2$ right angles.

But, it is given that

$$\angle ACD + \angle DCB = 2 \text{ right angles;}$$

$\therefore$ $\angle ACD + \angle DCB = \angle ACD + \angle DCE.$

From each of these equals, take the angle *ACD*:

$\therefore$ $\angle DCB = \angle DCE;$

$\therefore$ *CB* falls along *CE*.

But, by construction, *CA* and *CE* are the same straight line.

$\therefore$ *CA* and *CB* are in the same straight line.

*See 'NOTE' on page 5.

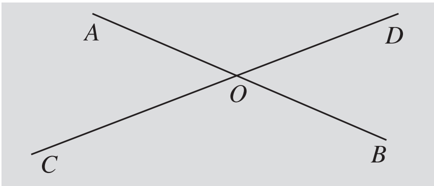
NOTE: In Theorem 1, it is given that *CA* and *CB* are in the same straight line, and it is proved that ∠ *ACD* + ∠ *DCB* = 2 rt. ∠s.

In Theorem 2, it is given that ∠ *ACD* + ∠ *DCB* = 2 rt. ∠s, and it is proved that *CA* and *CB* are in the same straight line.

These are therefore converse theorems (see p. 4).

THEOREM 3* (Euclid I. 15)

If two straight lines intersect, the vertically opposite angles are equal.



Let the straight lines *AB*, *CD* intersect at *O*. It is required to prove that

∠ *AOC* = ∠ *BOD* and ∠ *COB* = ∠ *DOA*.

Proof. Because *OC* stands on the straight line *AB*,

∴ ∠ *AOC* + ∠ *COB* = 2 right angles;

and because *OB* stands on the straight line *CD*,

∴ ∠ *BOD* + ∠ *COB* = 2 right angles;

∴ ∠ *AOC* + ∠ *COB* = ∠ *BOD* + ∠ *COB*.

From each of these equals take the angle *COB*;

∴ ∠ *AOC* = ∠ *BOD*.

Similarly it can be shown that

∠ *COB* = ∠ *DOA*.

EXERCISE I (on pages 1-7)

1. What is the angle in degrees between the hands of a watch at (*i*) 2 o'clock, (*ii*) 5 o'clock?
2. What angle does (*i*) the minute hand, (*ii*) the hour hand, turn through in 20 minutes?
3. What is the angle in degrees between the hands of a clock at (*i*) 12.15, (*ii*) 2.15 o'clock?
4. What are the supplements of 2/3rt. ∠, 40°, 120°, 98° 10′?
5. What are the complements of 1/4 rt. ∠, 40°, 35° 10′ 10″?
6. If in the figure of Theorem 1, the ∠ *BCD* = 32°, how large are the ∠s *ACD*, *ECD*?

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*See 'NOTE' on page 5.

7. If in the figure of the corollary of Theorem 1, $\angle AOB = 30°$, $\angle BOC = 70°$, $\angle COD = 140°$, what is the $\angle DOA$?

8. If in the figure of Theorem 3, the $\angle AOC = 42°$, what are the other angles in the figure?

9. A straight line *AB* is bisected at *C*, and produced to *D*. Prove that $DA + DB = 2DC$.

10. A straight line *AB* is bisected at *C*, and any point *D* is taken in *CB*. Prove that $AD - DB = 2CD$.

11. In the figure of Theorem 3, prove that the bisectors of the angles *AOD*, *DOB* are at right angles.

12. In the figure of Theorem 3, prove that the bisector of the *AOD*, when produced, bisects the $\angle BOC$.

13. In the figure of Theorem 3, prove that the bisectors of the angles *AOC*, *BOD* are in the same straight line.

14. *A*, *B*, *C*, *D* are four points, and *AB*, *BC* subtend (*i.e.* are opposite to) supplementary angles at *D*: show that *A*, *D*, *C* are in the same straight line.

15. *OA*, *OB*, *OC*, *OD* are straight lines so drawn that $\angle AOB = \angle COD$ and $\angle BOC = \angle AOD$. Show that *AO*, *OC* and also *BO*, *OD* are in the same straight line.

16. *CAD* and *AB* are two straight lines and $\angle CAX = \angle BAD$; *B*, *X* being on opposite sides of *C*, *D*. Prove that *AB*, *AX* are in the same straight line.

17. *XOA*, *XOB* are angles on the same side of *OX*, and *OC* bisects the angle *AOB*. Prove that $\angle XOA + \angle XOB = 2\angle XOC$.

18. *AOX*, *XOB* are adjacent angles, of which *AOX* is the greater, and *OC* bisects the angle *AOB*. Prove that $\angle AOX - \angle XOB = 2\angle COX$.

III. TRIANGLES

Def. A **plane figure** is any part of a plane surface bounded by one or more lines, straight or curved.

Def. A **rectilineal figure** is one which is bounded by straight lines.

It will be assumed that any figure may be **duplicated** (*i.e.* copied exactly), or that it may be moved from any one position to any other position, and, if necessary, *turned over* or *folded.*

If a figure is taken up and placed on another figure in order to make a comparison, the first figure is said to be applied to the second. This process is called **superposition**.

Figures which occupy the same portion of space are said to coincide.

Def. Figures which can be made to coincide are called **congruent**. Congruent figures are said to be **equal in all respects**.

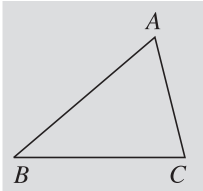
In two congruent figures, the sides and angles which coincide, when one figure is applied to the other, are said to correspond.

Def. The **area** of a plane figure is the amount of surface enclosed by its boundaries.

Def. **Equal figures** are those which have the same area.

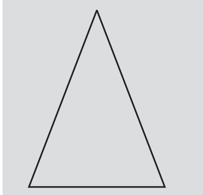
Def. The **perimeter** of a plane figure is the sum of the lengths of its boundaries.

Def. A **triangle** is a plane figure bounded by three straight lines.



The straight lines *BC*, *CA*, *AB* which bound a triangle *ABC* are called its sides and the points *A*, *B*, *C* its angular points or vertices.

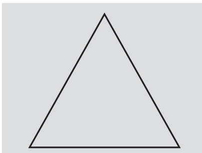
For distinction, one angular point is often called the vertex and the opposite side the base.



Def. An **isosceles triangle** is a triangle which has two equal sides.

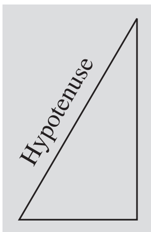
The side which is unequal to the others is always called the **base**, and the angular point opposite to the base is called the **vertex**.

Def. An **equilateral triangle** is a triangle whose three sides are equal.

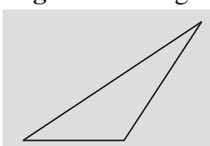


Def. A **right-angled triangle** is a triangle one of whose angles is a right angle.

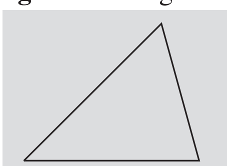
Def. The side opposite the right angle in a right-angled triangle is called the **hypotenuse**.



Def. An **obtuse-angled triangle** is a triangle which has one obtuse angle.

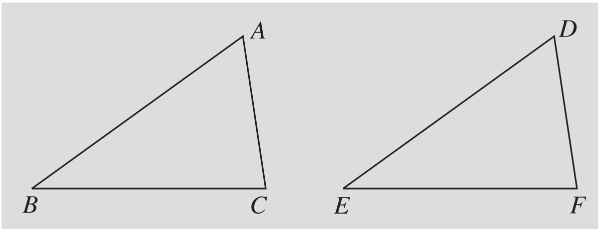


Def. An **acute-angled triangle** is a triangle which has three acute angles.



Theorem 4* (Euclid I. 4)

Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.



Let *ABC*, *DEF* be two triangles in which

$$AB = DE,$$
$$AC = DF$$

and the included $\angle A$ = The included $\angle D$.

It is required to prove that the triangles are congruent.

Proof. Apply the triangle *ABC* to the triangle *DEF* so that the point *A* falls on the point *D* and the straight line *AB* falls along the straight line *DE*.

Then because *AB* = *DE* (*given*),

∴ The point *B* falls on the point *E*.

Also since *AB* falls along *DE* and it is given that

$$\angle A = \angle D,$$

∴ *AC* falls along *DF*; and since it is given that

$$AC = DF.$$

∴ The point *C* falls on the point *F*.

Hence *A* falls on *D*, *B* on *E* and *C* on *F*.

∴ The triangle *ABC* has been made to coincide with the triangle *DEF*.

∴ The triangles *ABC*, *DEF* are congruent.

Note on Theorem 4. In the triangles *ABC*, *DEF*, *it is given that AB* = *DE*, *AC* = *DF*, included $\angle A$ = included $\angle D$; *it is proved that* the triangles are congruent, and therefore (*i*) the triangles are equal in area, (*ii*) *BC* = *EF*, (*iii*) $\angle B = \angle E$, $\angle C = \angle F$.

Observe that the angles which are proved equal are opposite equal sides in the two triangles.

In congruent triangles, the sides and angles which coincide, when one triangle is applied to the other, are said to correspond.

In using Theorem 4, the work should be written out as in the following example:

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*See 'Note' (italic) on page 5.